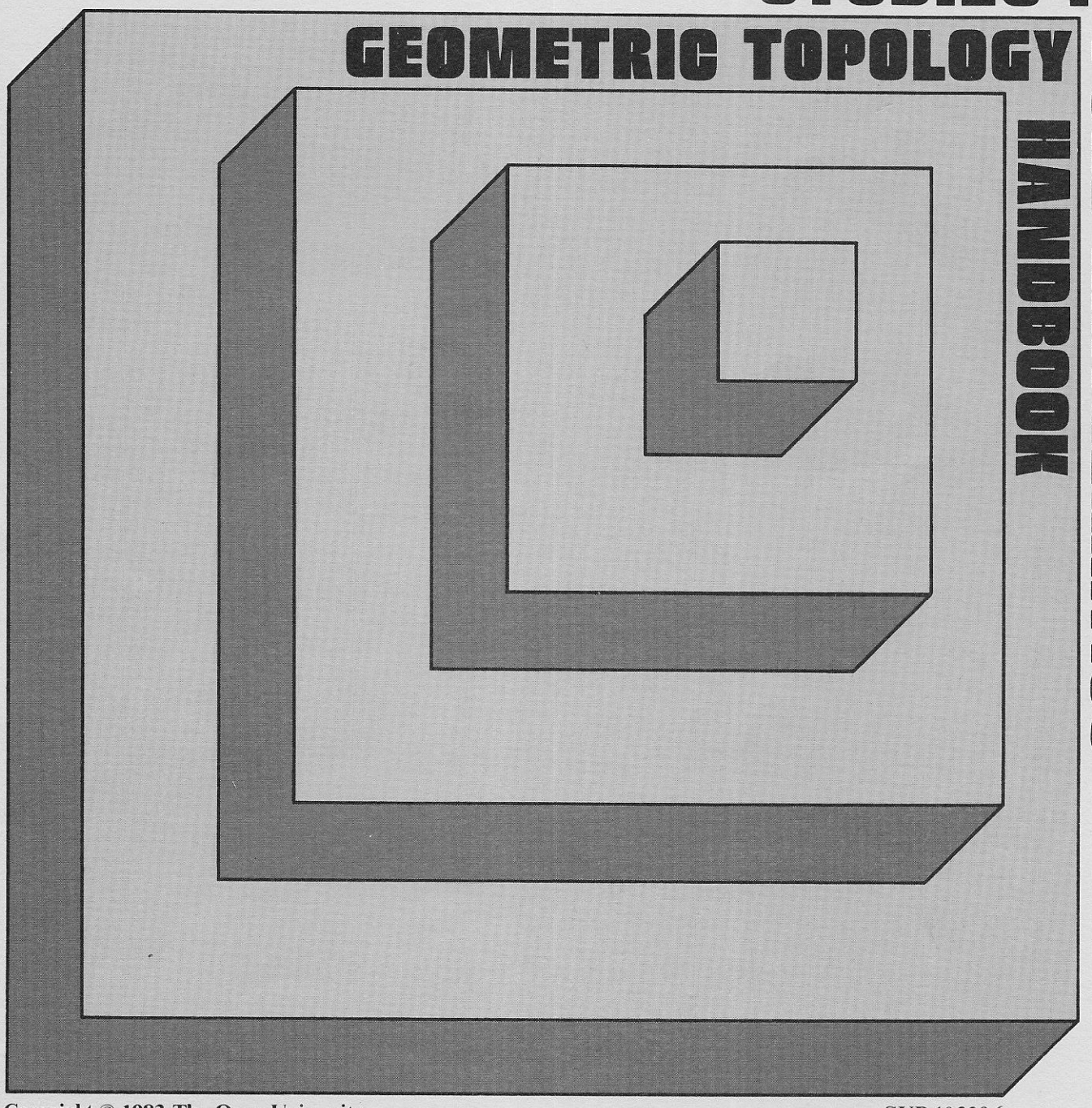
THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

Note that you may take this handbook into the examination at the end of the course, provided you have not made any written additions to it other than corrections.

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

HANDBOOK



Copyright © 1983 The Open University

SUP 10339 6

Unit 1 Surfaces

Polar coordinates

Polar coordinates (r, θ) in the plane are related to rectangular Cartesian coordinates (x, y) by the formulae

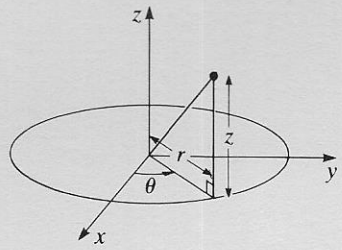
$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

B p. 3

Cylindrical coordinates

Replacing x and y in the Cartesian coordinates (x, y, z) by polar coordinates gives cylindrical coordinates (r, θ, z) (also known as cylindrical polar coordinates).

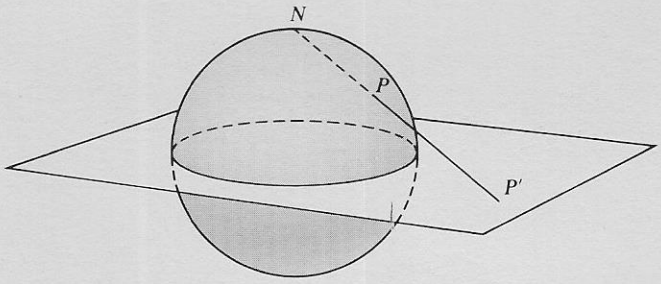
B pp. 3, 4



Stereographic projection

The one-to-one correspondence of the unit sphere $x^2 + y^2 + z^2 = 1$ minus the "north pole" $N = (0, 0, 1)$, with the equatorial plane $z = 0$, which maps the point P of the sphere to the point P' where the line PN meets the plane $z = 0$.

B p. 10



Homeomorphism

A continuous function $f: X \longrightarrow Y$ with an inverse function $f^{-1}: Y \longrightarrow X$ which is also continuous.

Unit 1, Section 1.3
B pp. 12, 13

Topological sphere

Any set homeomorphic to the standard sphere in $\mathbb{R}^3$, $\{(x, y, z): x^2 + y^2 + z^2 = 1\}$.

B p. 15

Topological torus

Any set homeomorphic to a standard torus (e.g. the one obtained by rotating a circle radius 1 and centre (0, 2, 0) lying in the yz-plane about the z-axis).

B p. 14

Path-connected surface (or set)

A surface S (or, in general, any set) is path-connected if, given any two points A, B of S, there is a path in S from A to B; that is, there is a continuous function, or path,

$$f: [0, 1] \longrightarrow S \quad \text{with } f(0) = A \text{ and } f(1) = B.$$

Unit 1, Section 1.3

Closed rectangle

A closed rectangle in $\mathbb{R}^2$ is a set

$$\{(x, y): a \leq x \leq b,\ c \leq y \leq d\}.$$

Unit 1, Section 1.3

Closed set

A set $X \subseteq \mathbb{R}^2$ is closed if every point c *not* in X can be "closed off" by a closed rectangle containing c in its inside (not on the boundary) which does not meet X.

Unit 1, Section 1.3

Compact set

A set which is both closed and bounded.

Unit 1, Section 1.3

Neighbourhood

A subset of a surface S is a neighbourhood of $P \in S$ if it contains all the points of S within a small distance of P. (The neighbourhood is allowed to contain other points as well.)

Locally disc-like A set S is "locally disc-like" if, for every point P of S, we can find a homeomorphism, f_P, from a neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$ carrying P to $(0, 0)$. *Unit 1*, Section 1.4

Surface without boundary A set S is a surface without boundary if: *Unit 1*, Section 1.4

(i) S is locally disc-like;

(ii) S is path-connected;

(iii) S is homeomorphic to a compact subset of $\mathbb{R}^n$.

Surface with boundary As above, replacing (i) by

(i)′ for each point P we can find

either (a) a homeomorphism f_P from a neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$, carrying P to $(0, 0)$,

or (b) a homeomorphism g_P from a neighbourhood of P to the halfdisc $\{(x, y): x^2 + y^2 < 1, y \geq 0\}$, carrying P to $(0, 0)$.

Subdivision of a surface A subdivision of a surface is a dissection of the surface into vertices, edges and faces satisfying the following. *Unit 1*, Section 1.4

1. Each edge starts or ends at a vertex.
2. Edges cross or meet only at vertices.
3. The edges form a path-connected network.
4. The faces are topological discs (homeomorphic to the standard disc).

Boundary of a surface The subset of points P of the surface where *no* homeomorphism exists of the type f_P given in the alternative condition (i)′ (a) above, so that only a homeomorphism g_P of a neighbourhood with a halfdisc exists.

Euler characteristic The Euler characteristic of a (subdivided) surface S with F faces, E edges and V vertices is $\chi(S) = F - E + V$.

Theorem (Not proved in the course) *Unit 2*, Section 2.2
The boundary of a surface consists of a finite number of disjoint curves, each homeomorphic to a circle. Each curve is known as a *boundary curve* of the surface, and the number of boundary curves is a topological invariant.

Classification Theorem A surface is determined up to homeomorphism once we know:

its Euler chacteristic;

how many boundary curves it has;

whether or not it is orientable ("orientable" is defined in *Unit 2*).

Unit 2 Subdivisions and Orientability

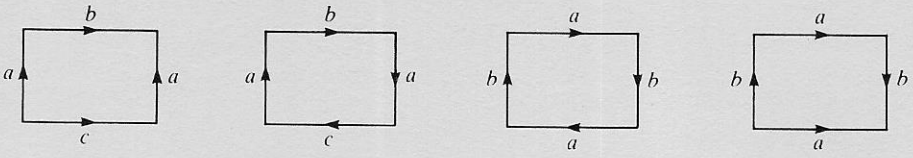
Triangulation A triangulation of a surface is a subdivision (see above) also satisfying the following. *Unit 2*, Section 2.1

1. Every face has exactly 3 edges and 3 vertices.
2. Every edge has exactly 2 vertices.
3. There is at most one face with 3 given vertices.
4. There is at most one edge with 2 given vertices.

Regular subdivision A subdivision in which each face has the same number of edges, and each vertex is at the end of the same number of edges. *Unit 2*, Section 2.1

Cylinder (or annulus)
Möbius band
Projective plane
Klein bottle

Surfaces which can be made by identifying edges of a rectangle, as follows. *Unit 2*, Section 2.3



Cylinder Möbius band Projective plane Klein bottle

Crosscap A name for a particular representation of the Möbius band in $\mathbb{R}^2$ which has self-intersections. *Unit 2*, Section 2.3

Orientation An *orientation* of a surface is any process or rule which defines a consistent sense of rotation over the whole surface. *Unit 2*, Section 2.4

Orientable and non-orientable A surface is called *orientable* if it can be given an orientation, and *non-orientable* otherwise. *Unit 2*, Section 2.4

enting a face of a subdivision To specify an orientation of a face of a subdivision, it is necessary only to specify a direction of travel around the boundary. *Unit 2*, Section 2.4

Compatibility of ntation of faces In a subdivided surface, we say that the orientations of two faces having a common edge b are compatible across b if the "arrows" showing the direction of travel round the boundaries of the two faces are in opposite directions on the two sides of b. *Unit 2*, Section 2.4

Orientation of a bdivided surface An orientation of a subdivided surface is an orientation of each face which satisfies the compatibility condition above. *Unit 2*, Section 2.4

Theorem A surface is non-orientable if and only if it contains a Möbius band. *Unit 2*, Section 2.4

Theorem A non-orientable surface without boundary cannot be embedded in $\mathbb{R}^3$. *Unit 2*, Section 2.4

Unit 3 Combinatorial Surfaces

Let D be a disc with n evenly spaced vertices $p_1, \ldots, p_n$ which divide the boundary circle of D into n arcs $a_1, \ldots, a_n$.

(n-sided) logical polygon An (n-sided) topological polygon π is the image of D under a one-to-one continuous function f. The points $f(P_i)$ are called the *vertices* of π, and the (curved) lines $f(a_i)$ are called the *edges* of π.

Singular logical polygon A singular topological polygon is the image of D under a continuous function f which is one-to-one except possibly on the boundary of D. For each edge a_i of D, there are two possibilities. For each point P of a_i other than the two vertices P_i, P_{i+1} (interpret P_{n+1} as P_1):

(i) there is no other point Q of D with $f(P) = f(Q)$;

or

(ii) there is exactly one point P' with $f(P') = f(P)$, and in this case as P moves from P_i to P_{i+1}, P' also traces out another edge a_j, either from P_j to P_{j+1} or from P_{j+1} to P_j.

We say the edge $b_i = f(a_i)$ is identified with the edge b_j (if P' traces out a_j from P_j to P_{j+1}) or with b_j^{-1} (if P' traces out a_j from P_{j+1} to P_j).

Edge equation

The edge equation of a (singular) topological polygon is obtained from $b_1, \ldots, b_n = 1$ as follows.

Replace b_j with b_i if b_j is identified with b_i for $i < j$, and replace b_j with b_i^{-1} if b_j^{-1} is identified with b_i for $i < j$.

Combinatorial representation of a surface

A system of edge equations is a combinatorial representation of a surface such that:

B p. 61

(i) no letter appears more than twice;

(ii) no matter how the system is divided into two sets of equations, at least one letter appears in an equation of each set.

Trivial equivalence of edge equations

An equivalence achieved by a sequence of operations of types 1 and 2 below.

1. An initial (terminal) letter transposed to become the initial (terminal) letter on the opposite side of an equation, for example
$$aB = C \longrightarrow B = a^{-1}C \quad \text{or} \quad Ba = C \longrightarrow B = Ca^{-1},$$
where B, C stand for any blocks of letters.
2. A new letter, not appearing in the equations, is substituted for an old letter wherever the old letter appears.

"Cut and paste" operations or equivalences

1. Select a new letter, x, and replace an equation $ABC = 1$ by the two equations $AxC = 1$ and $x = B$ (where A, B, C stand for any blocks of letters).
2. Select a new letter, x, and insert the combination xx^{-1} (or $x^{-1}x$) in any equation in any position.
3. Select new letters, u, v, and replace an existing letter x with uv (and x^{-1} by $v^{-1}u^{-1}$, if x^{-1} occurs).
4. The inverses of 1, 2, 3.

Unit 3, Section 3.

Combinatorial equivalence

Two systems of edge equations are combinatorially equivalent if one system can be transformed into the other by a sequence of trivial equivalences and "cut and paste" equivalences, in any order.

Unit 3, Section 3.

Theorem

(Not proved in the course)
Combinatorially equivalent surfaces are homeomorphic.

Models of $\mathbb{RP}^2$

There are various ways of constructing the projective plane. All the following constructions, or "models", are homeomorphic.

Geometer's "plane" model

The "points" are

(i) ordinary points of $\mathbb{R}^2$,

or

(ii) "ideal" points—one ideal point for every direction in $\mathbb{R}^2$.

Geometer's "ray" model

The "points" are lines through the origin in $\mathbb{R}^3$.

"Sphere" model

The "points" are *pairs* of antipodally opposite points on S^2, the standard unit sphere.

"Hemisphere" model

The southern hemisphere of S^2, with the antipodally opposite points on the equator identified.

"Topologist's" model

A square with the edges identified according to the edge equation $abab = 1$.

Theorems

1. The projective plane is non-orientable and consists of a Möbius band glued to a disc along their boundary curves ∂:
$$\mathbb{RP}^2 = M^2 \cup_\partial D^2.$$
2. A projective line is a topological circle, and since it is the central circle of a Möbius band, it is orientation-reversing. A conic is also a topological circle, but since it is the central circle of an annulus, it is not orientation-reversing.

Unit 4 Canonical Equations and Connected Sums of Surfaces

Handle When an edge equation has a block of symbols of the form $aba^{-1}b^{-1}$, we refer to this as a *handle*. It corresponds geometrically to a torus with a hole or "handle". *Unit 4*, Section 4.1 See also Section 4.3

Crosscap A block of symbols aa in an edge equation *or* a particular representation of the Möbius band which corresponds geometrically to such a block.

Cuff A block of symbols ded^{-1} in an edge equation.

Canonical equation An edge equation of the form *Unit 4*, Section 4.2

$$\underbrace{(a_1b_1a_1^{-1}b_1^{-1}) \ldots (a_pb_pa_p^{-1}b_p^{-1})}_{p \text{ blocks}} \underbrace{d_1e_1d_1^{-1} \ldots d_re_rd_r^{-1}}_{r \text{ blocks}} = 1$$

(orientable canonical form),

or

$$\underbrace{a_1a_1 \ldots a_qa_q}_{q \text{ blocks}} \underbrace{d_1e_1d_1^{-1} \ldots d_re_rd_r^{-1}}_{r \text{ blocks}} = 1$$

(non-orientable canonical form).

Genus

(a) *Orientable surface*: the number of handles in the canonical equation.

(b) *Non-orientable surface*: the number of crosscaps in the canonical equation.

Vertex symbol (xy) Given an edge equation $AxyB = 1$, the vertex symbol (xy) is used for the common vertex of the two edges x and y, i.e. the terminal vertex of x and the initial vertex of y. Similarly, $(v^{-1}w)$ is the common vertex of v^{-1} and w, i.e. the terminal vertex of v^{-1} (=initial vertex of v) and the initial vertex of w.

Connected sum of two surfaces To form the connected sum $L \# M$ of two surfaces L and M, remove an (open) disc from each and glue L to M along the circular boundaries of the holes thus created. *Unit 4*, Section 4.3

Lemmas on symbol manipulation

1. If a polygon of a surface has an equation *Unit 4*, Section 4.1
$$AxBCxD = 1,$$
it is combinatorially equivalent to the surface with this equation replaced by
$$AxCxB^{-1}D = 1,$$
and vice versa.
2. Similarly, combinatorial equivalence permits an equation of the form $AxBCx^{-1}D = 1$ to be replaced by $AxCBx^{-1}D = 1$.
3. Any system of edge equations is combinatorially equivalent to a system with a single equation. Hence, any surface may be represented by a single equation.

Theorem S is non orientable in the *Unit 2* sense $\iff$ S is represented by a single edge equation of the form $\ldots a \ldots a \ldots = 1$. *Unit 4*, Section 4.2

Theorem Orientability, Number of boundary curves, Euler characteristic — are all preserved by combinatorial equivalence. *Unit 4*, Section 4.2

Theorem Every system of edge equations is combinatorially equivalent to one of the canonical quations. Every surface is combinatorially equivalent to, and hence homeomorphic to, one of the surfaces given by a canonical equation. More explicitly, an orientable surface must be a sphere with p handles and r cuffs, and a non-orientable surface must be a sphere with q crosscaps and r cuffs. *Unit 4*, Section 4.2

Theorem $\mathbb{RP}^2$ minus an open disc is homeomorphic to a Möbius band, M^2: *Unit 4*, Section 4.3

$$\mathbb{RP}^2 \backslash \mathring{D}^2 \cong M^2.$$

Theorem The Klein bottle, K^2, is homeomorphic to two Möbius bands glued along their boundaries:

$$K^2 \cong M^2 \cup_\partial M^2.$$

Theorem

$$3\,\mathbb{RP}^2 \cong T^2 \# \mathbb{RP}^2$$

Theorem For any surface S,

$$S \# S^2 \cong S, \qquad S \# D^2 \cong S \backslash \mathring{D}^2.$$

Theorem An orientable surface is either the sphere S^2 or else is homeomorphic to the connected sum of p tori and r discs; in symbols, $S = S^2$ or $S \cong pT^2 \# rD^2$. *Unit 4*, Section 4.3

A non-orientable surface S is homeomorphic to the connected sum of q projective planes and r discs; in symbols, $S \cong q\mathbb{RP}^2 \# rD^2$.

Theorem If two surfaces, L, M, have canonical equations $A = 1$, $B = 1$, respectively, then $AB = 1$ is an edge equation for $L \# M$. *Unit 4*, Section 4.3

Theorem (Not proved in the course)
Every surface can be triangulated.

Theorem (Not proved in the course)
The Euler characteristic is a topological invariant of a surface; that is, if L, M are homeomorphic surfaces, then $\chi(L) = \chi(M)$.

Unit 5 Complex Loci and the Riemann Sphere

Complex locus The complex locus $f(z, w) = 0$ is the set of pairs of complex numbers (z, w) which satisfy the equation. *Unit 5*, Section 5.1

Riemann sphere The plane of complex numbers plus a "point at infinity", usually denoted by ∞. The Riemann surface itself is denoted by $\mathbb{C} \cup \{\infty\}$. Stereographic projection gives a one-to-one correspondence from the sphere S^2 to $\mathbb{C} \cup \{\infty\}$, with the north pole N corresponding to ∞. *Unit 5*, Section 5.1

"Infinite points" of a complex locus Given a complex locus $f(z, w) = 0$, where f is a polynomial in z and w, we say: *Unit 5*, Section 5.1

(i) a point (∞, w), $w \neq \infty$, satisfies $f(z, w) = 0$ if and only if $(\tilde{z}, w) = (0, w)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, w\right) = 0$;

(ii) a point (z, ∞), $z \neq \infty$, satisfies $f(z, w) = 0$ if and only if $(z, \tilde{w}) = (z, 0)$ satisfies the transformed equation $f\left(z, \frac{1}{\tilde{w}}\right) = 0$;

(iii) the point (∞, ∞) satisfies $f(z, w) = 0$ if and only if $(\tilde{z}, \tilde{w}) = (0, 0)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, \frac{1}{\tilde{w}}\right) = 0$.

In all these cases, the transformed equation is obtained by substituting $\frac{1}{\tilde{z}}$ or $\frac{1}{\tilde{w}}$ for z or w (or both), and, if necessary, multiplying the result by a power of $\tilde{z}$ or $\tilde{w}$ to get a polynomial equation. Points of the form (z, ∞), (∞, w), (∞, ∞), are called infinite points of the locus.

Complex conic A complex conic is the complex locus $f(z, w) = 0$, where the polynomial f has degree 2 in z and w, i.e.

Unit 5, Sections 5.2, 5.3

$$f(z, w) = Az^2 + Bzw + Cw^2 + Dz + Ew + F,$$

where A, B, C, D, E, F are complex numbers, not all of A, B and C are zero, and z, w are allowed to take complex values.

Non-degenerate complex conic $f(z, w) = 0$ is *degenerate* if f factors into two linear polynomials, and is non-degenerate otherwise.

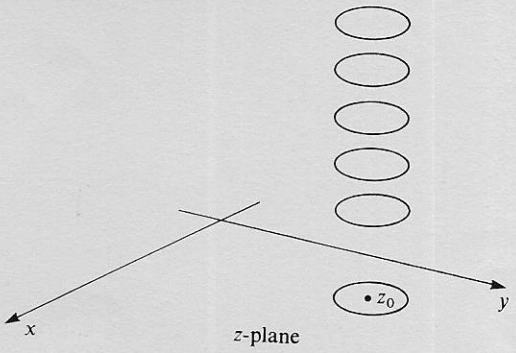
Unit 5, Section 5.3

***n*-fold cover** Given a complex locus

Unit 5, Section 5.1

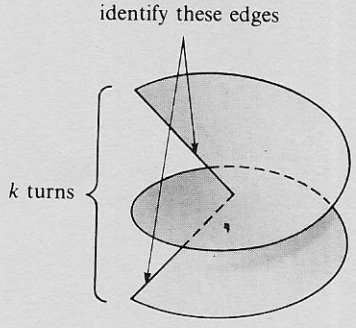
$$f(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0$$

(where each $a_i(z)$ is a polynomial in z), then, for each fixed z_0, $f(z_0, w)$ is a polynomial in w of degree at most n. Suppose, for a particular value of z_0, $f(z_0, w) = 0$ has n distinct roots $w(z_0)$.

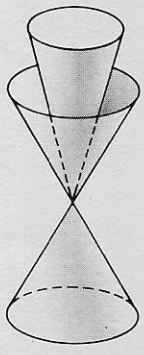


If z_0 varies over a small disc in the z-plane, each of the values of $w(z_0)$ varies over a small disc, which we draw in $\mathbb{R}^3$ as if they are sitting over the patch containing z_0. We say the locus is an *n-fold cover* if $f(z_0, w) = 0$ has exactly n roots, except possibly at a finite number of points z_0.

Pinch points, Branch points As z varies over the whole of $\mathbb{C} \cup \{\infty\}$, the discs referred to in the definition of an n-fold cover (given above) join up at the exceptional points z_0 where $w(z_0)$ has *fewer* than n distinct values. Two ways in which they join up for the special values of z are *pinch points* and *branch points*.



Branch points



Pinch points

Note that, strictly speaking, branch points and pinch points are points on the *locus*, not the values of z where they occur.

Riemann surface Suppose $f(z, w) = Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$ is the equation of a non-degenerate complex conic. For a fixed value of z, the equation has two distinct roots, except where

$$(Bz + E)^2 - 4C(Az^2 + Dz + F) = 0. \qquad (*)$$

Unit 5, Section 5.3

B pp. 90–3

Three cases occur.

1. $C = 0$. There is only one value of w for each value of z, and the Riemann surface is the locus itself (plus the infinite point).
2. $B^2 - 4AC \neq 0$. The equation $(*)$ has two distinct roots, t_1, t_2. If we cut $\mathbb{C} \cup \{\infty\}$ along an edge a from ∞ to t_1, and along an edge b from ∞ to t_2, then the Riemann surface consists of two sheets, each equal to the cut sphere, joined according to the edge equations
$$a_1 a_2^{-1} b_1 b_2^{-1} = 1, \quad a_2 a_1^{-1} b_2 b_1^{-1} = 1,$$
where the suffix 1 refers to the first sheet and the suffix 2 refers to the second sheet.
3. $B^2 - 4AC = 0$. The equation $(*)$ has one root, t_1, and the Riemann surface consists of two sheets, each obtained by cutting the Riemann sphere from t_1 to ∞ along a and joining the sheets according to the equations $a_1 a_2^{-1} = 1$, $a_2 a_1^{-1} = 1$.

Theorem A non-degenerate complex conic

$$Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$$

(plus its infinite points) is a sphere if $A = 0$ or $C = 0$ or $B^2 - 4AC = 0$, and is a pinched sphere otherwise. The branch points and pinch points occur as follows.

Unit 5, Section 5.3

1. $A = 0$, $C \neq 0$. There are two branch points, one of which is (∞, ∞) if $B = 0$, and no pinch points.
2. $C = 0$. No branch points, no pinch points.
3. $A \neq 0$, $C \neq 0$, $B^2 - 4AC = 0$. Two branch points, one of which is (∞, ∞).
4. Two branch points (not at infinity), and a pinch point at (∞, ∞).

Unit 6 Covering Spaces

Local homeomorphism A continuous function $p: \tilde{S} \longrightarrow S$ between two surfaces is a local homeomorphism if every $\tilde{P} \in \tilde{S}$ has a (small) disc $D_{\tilde{P}}$ round $\tilde{P}$ with the property that p restricted to $D_{\tilde{P}}$ is a homeomorphism.

Unit 6, Section 6.1

Regular n-fold covering surface (also called an unbranched n-fold covering surface) If we have a local homeomorphism $p: \tilde{S} \longrightarrow S$ between two surfaces, and the inverse image of every $P \in S$ consists of exactly n points of $\tilde{S}$, we say that $p: \tilde{S} \longrightarrow S$ is a regular (or unbranched) n-fold covering surface. S is called the base surface and $\tilde{S}$ is often (loosely) referred to as the covering surface, or sometimes the "total space".

n-sheeted covering surface Starting with any set of edge equations representing a surface S, make n copies of S and its equations, called the *sheets* of the covering, by writing down the original equations n times, with subscripts 1 to n added to the edge symbols to denote the copy of S to which it belongs. If an edge symbol occurs twice in the original set of equations, permute the subscripts on the second occurrences of the edge symbol in the n copies, in any manner, provided that the combined sets of equations result in a surface $\tilde{S}$. This joins up the n sheets of the covering to make the surface $\tilde{S}$. $\tilde{S}$ is called an *n-sheeted covering surface of S.*

Unit 6, Section 6.2

Riemann surface If S is the Riemann sphere, $\tilde{S}$ is called a *Riemann surface.*

Order of a branch point = branch point deficit

Suppose $p:\tilde{S} \longrightarrow S$ is an n-sheeted covering surface and $\tilde{P}$ is a branch point of $\tilde{S}$. Then at $\tilde{P}$, $k > 1$ sheets of the covering surface are joined together so that instead of one point over $P = p(\tilde{P})$ for each of the k sheets, there is only the point $\tilde{P}$. Then the deficiency in the number of covering points $k - 1$ is called the *order* of the branch point $\tilde{P}$.

Unit 6, Sections 6.2, 6.3

Total branch deficit

The total of the orders (or deficits) of each branch point.

Unit 6, Section 6.2

Theorem

Suppose $p:\tilde{S} \xrightarrow{n} S$ is an n-sheeted (n-fold) covering surface; then

Unit 6, Section 6.2

(i) if S has β boundary curves, $\tilde{S}$ has at least β and at most $n\beta$ boundary curves—in particular, if $\tilde{S}$ is closed (no boundary curves) then so is S, and vice versa; Section 6.3

(ii) if S is orientable, so is $\tilde{S}$; Section 6.2

(iii) $\chi(\tilde{S}) = n\chi(S) - \delta$, where δ is the total branch deficit—in particular, $\chi(\tilde{S}) = n\chi(S)$ for an *unbranched* cover. Section 6.2

Unit 7 Winding Numbers and Brouwer Degree

Loop

A loop in the plane (in a surface S) is a continuous function $f\colon [a, b] \longrightarrow \mathbb{R}^2(S)$.

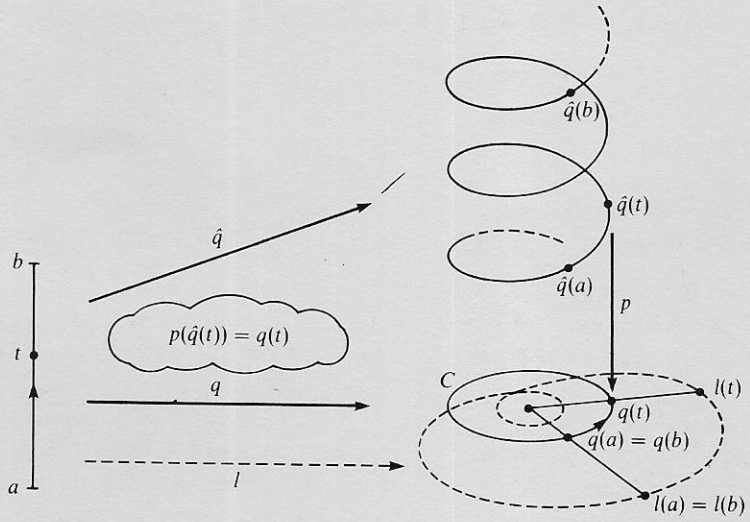
Unit 7, Section 7.1

Covering space of a circle

The continuous function $p\colon \mathbb{R} \longrightarrow C$ exhibits $\mathbb{R}$ as a covering space of the circle C if p is a local homeomorphism; that is, if every $x \in \mathbb{R}$ has a small interval about x on which p is a homeomorphism.

Winding number

In the diagram, l is a loop, $\mathbb{R}$ is a spiral with unit distance between successive turns, and $p\colon \mathbb{R} \longrightarrow C$ is a covering space of the circle C.



Define a function $q\,[a, b] \longrightarrow C$, the "circle of directions" centre P, by letting $q(t)$ be the point where the ray (halfline) from P to $l(t)$ cuts C. Choose a point $\hat{q}(a)$ in the spiral $\mathbb{R}$ which projects to $q(a)$, and lift q to a continuous map $\hat{q}\colon [a, b] \longrightarrow \mathbb{R}$ so that $p(\hat{q}(t)) = q(t)$ for all t in $[a, b]$. Then the winding number of l about P is the vertical distance between $\hat{q}(a)$ and $\hat{q}(b)$.

Homotopy of loops

The loop l_1 is homotopic to the loop l_2 if we can find a family of loops l_v, one loop l_v for each v between 1 and 2, such that the family gives us a continuous variation of l_1 into l_2. That is, if v and w are close, the image points of l_w and l_v are close; in other words, for each point t in $[a, b]$, $l_v(t)$ is close to $l_w(t)$.

Unit 7, Section 7.1

Multiplicity Suppose S, T are orientable surfaces, and $f: S \longrightarrow T$ is a continuous function with the additional property that only a finite number of points of S are mapped to any one point of T (that is, $f^{-1}(Q)$ is a finite set of points of S for every $Q \in T$). The multiplicity of a point $P \in S$ is the winding number of $f(l)$ about $f(P)$, where l is a small loop winding once positively round P. ("Small" means l contains no points of S mapping to $f(P)$ other than P itself.)

(This is also known as the *geometric multiplicity* of P, to distinguish it from the algebraic multiplicity defined later.)

Brouwer degree See Brouwer Degree Theorem.

Argument of a complex number z The complex number $z = x + iy$ (x, y real numbers) can be written in the form $z = r(\cos\theta + i\sin\theta)$. θ is called the angular coordinate or argument of z. It is written $\arg z$ for short.

Variation in $\arg z$ on a loop l If $l: [a, b] \longrightarrow \mathbb{C}$ is a curve (not necessarily a loop) in the plane of complex numbers $\mathbb{C}$, and can be written in the form

$$l(t) = r(t)[\cos\theta(t) + i\sin\theta(t)],$$

where r, θ are *continuous* functions of t, then the variation in $\arg z$ on l is given by $\text{var}(\arg l(t)) = \theta(b) - \theta(a)$.

Zero and pole of a rational function Suppose $f: \mathbb{C} \longrightarrow \mathbb{C}$ is a rational function, that is, $f(z) = p(z)/q(z)$ where p, q are polynomials. Assume p, q have no common factor. Then a *zero* of f is a point where $p(z) = 0$, and a *pole* of f is a point where $q(z) = 0$. These terms can be extended to functions $f: \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$; a zero of f is then a point which maps to 0 and a pole is a point which maps to ∞. *Unit 7*, Section 7.4

Algebraic multiplicity of a zero (pole) If $f(z) = p(z)/q(z)$, where p, q are polynomials with no common factors, then the *algebraic multiplicity* of a zero a of f is the integer r where $p(z) = (z - a)^r \bar{p}(z)$ and $\bar{p}(a) \neq 0$. A similar definition applies for a pole, with q replacing p.

Theorem If l_1 and l_2 are homotopic loops, and P does not lie on any of the intermediate loops l_n of the homotopy, then l_1 and l_2 have the same winding number about P. *Unit 7*, Section 7.1

Theorem Given a loop in the plane, the loop divides the plane into regions such that any point in a region can be reached from any other point in the same region without crossing the loop. Then *Unit 7*, Section 7.1

1. the winding numbers of the loop about two points in the same region are the same;
2. the winding number increases by one when crossing a loop segment traced out from left to right.

Theorem (Brouwer Degree Theorem) Given a continuous function $f: S \longrightarrow T$ between closed orientable surfaces such that every point Q in T has a finite number of pre-images (points P mapping to Q), define *Unit 7*, Section 7.1

$$N_Q = \text{sum of multiplicities of all pre-images of } Q.$$

Then N_Q is the same for every point Q of T. This common value is called the Brouwer degree of f.

Theorem If $f: [a, b] \longrightarrow \mathbb{C}$ is a loop, and continuous functions r, θ are given so that *Unit 7*, Section 7.3

$$l(t) = r(t)[\cos\theta(t) + i\sin\theta(t)],$$

then the winding number of l about 0 is $\dfrac{1}{2\pi}(\theta(b) - \theta(a))$.

Theorem Suppose $f, g: [a, b] \longrightarrow \mathbb{C}$ are loops, and the loop $l: [a, b] \longrightarrow \mathbb{C}$ is defined by the formula $h(t) = f(t)g(t)$. Then, about any point not on the loops, *Unit 7*, Section 7.3

$$\omega(h) = \omega(f) + \omega(g).$$

Theorem (Rouché) Suppose l is a loop in $\mathbb{C}$ and $f, g: \mathbb{C} \longrightarrow \mathbb{C}$ are two continuous functions satisfying *Unit 7*, Section 7.3

$$|f(P)| < |g(P)|$$

for all points P on the loop l. Then the winding numbers about 0 satisfy

$$\omega(g(l)) = \omega(h(l)), \qquad \text{where } h \text{ is the sum function } f + g.$$

Unit 8 Poincaré Index Theorem

Flow field = vector field A function V which assigns to each point (x, y) of $\mathbb{R}^2$ a vector $V(x,y)$ based at (x, y). In coordinates, we write *Unit 8*, Section 8.1

$$V(x, y) = (f(x, y), g(x, y)).$$

We assume that the functions f, g have continuous partial derivatives throughout this unit, without stating it every time.

Linear flow field The functions f, g are linear functions *Unit 8*, Section 8.1

$$f(x, y) = ax + by, \quad g(x, y) = cx + dy$$

(a, b, c, d real numbers).

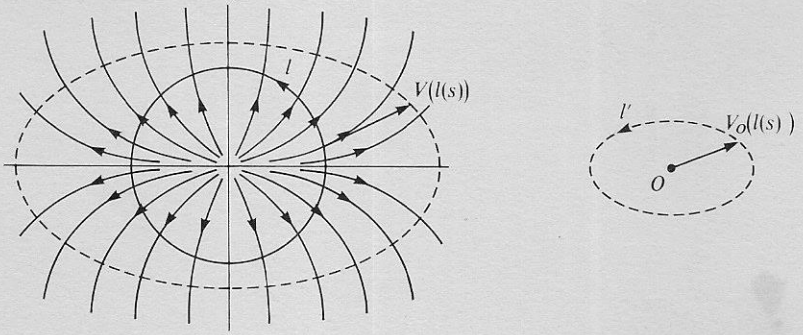
Rest point = singular point = singularity A point where $V(x, y) = (0, 0)$.

Unstable rest point A rest point such that any point close to the rest point "flows away" to infinity as t increases.

Stable rest point A rest point such that any point close to the rest point remains close as t increases.

Isolated rest point = isolated singular point A rest point so that no other nearby point is a rest point.

Index of an isolated singular point Draw a small simple loop l (say the boundary of a small disc), centre P, which winds once positively about P. At each point $l(s)$ of the loop, there is therefore a non-zero flow vector $V(l(s))$. Let $V_O(l(s))$ be a vector equal to $V(l(s))$, but based at some fixed point O. The "sharp end" of $V_O(l(s))$ will trace out a loop l' as $l(s)$ traces out l. The index of P is the winding number of l' about O. *Unit 8*, Section 8.1



This definition allows P to be a non-singular point where $V(P) \neq 0$—the index is then zero—but it does *not* allow P to be a point on a line of singular points. *Unit 8*, Section 8.1

Theorem (Poincaré Index Theorem) Let S be any closed orientable surface, and V a flow on S with only isolated singularities. Then the sum of the indices of all singular points of V is equal to $\chi(S)$.

Corollaries

1. The torus is the only closed orientable surface which has a smooth flow without singular points.
2. (Hairy Ball Theorem)
   There is no non-singular flow on S^2.

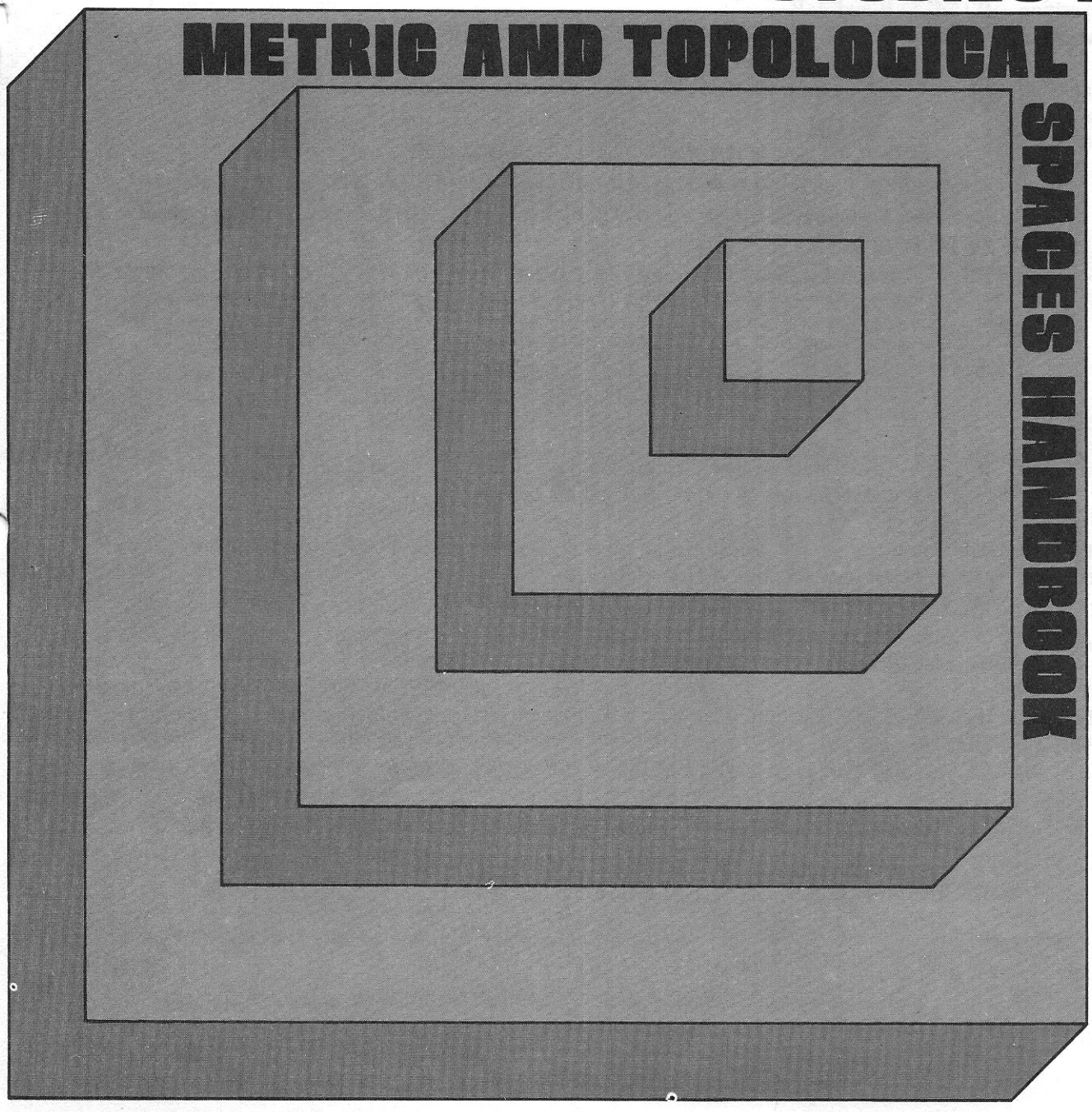
Produced in Great Britain by Speedlith Photo Litho Limited, Manchester M32 0JT.

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

Note that you may take this handbook into the examination at the end of the course, provided you have not made any written additions to it other than corrections.

STUDIES IN PURE MATHEMATICS

METRIC AND TOPOLOGICAL SPACES HANDBOOK



Copyright © 1983 The Open University

SUP 09762 4

Unit 1 Sets and Functions

Union and intersection

The union of an indexed family $(A_i)_{i\in I}$ of sets is the set

$$\bigcup_{i\in I} A_i = \{x : x \in A_i \text{ for } \textit{some } i \text{ in } I\};$$

and the intersection of the family $(A_i)_{i\in I}$ is the set

$$\bigcap_{i\in I} A_i = \{x : x \in A_i \text{ for } \textit{every } i \text{ in } I\}.$$

Definitions, page 14

Complement

If A and B are sets, then the relative complement of B in A is the set

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

If $B \subset A$, then $A - B$ is called the complement of B in A.

Page 10

De Morgan's Laws

If B is a set and $(A_i)_{i\in I}$ is an indexed family of sets, then:

(i) $B - (\bigcup_{i\in I} A_i) = \bigcap_{i\in I} (B - A_i)$;

(ii) $B - (\bigcap_{i\in I} A_i) = \bigcup_{i\in I} (B - A_i)$.

Proposition 3, page 16

Power set

The power set of a set A, denoted by 2^A, is the set whose elements are the subsets of the set A, so that $2^A = \{B : B \subset A\}$.

Page 9

Composition of functions

If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions, then the composition of f and g is the function $g \circ f : A \longrightarrow C$ given by $(g \circ f)(a) = g(f(a))$, $a \in A$.

Page 18

Image and inverse image

Let $f : A \longrightarrow B$ be a function. If $C \subset A$, then the image of C under f is the set

$$f(C) = \{b \in B : b = f(a) \text{ for some } a \text{ in } C\};$$

if $D \subset B$, then the inverse image of D under f is the set

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

Page 18

If $f : A \longrightarrow B$ is a function, $C \subset A$ and $D \subset B$, then $C \subset f^{-1}(D)$ if and only if $f(C) \subset D$.

Proposition 4, page 18

If $f : A \longrightarrow B$ is a function, $C \subset A$ and $D \subset B$, then

$$C \subset f^{-1}(f(C)) \quad \text{and} \quad f(f^{-1}(D)) \subset D.$$

SAQ 14, page 23

If $f : A \longrightarrow B$ is a function and $(D_j)_{j\in J}$ is an indexed family of subsets of B, then

$$f^{-1}\left(\bigcup_{j\in J} D_j\right) = \bigcup_{j\in J} f^{-1}(D_j) \quad \text{and} \quad f^{-1}\left(\bigcap_{j\in J} D_j\right) = \bigcap_{j\in J} f^{-1}(D_j).$$

SAQ 15, page 24

Restriction

If $f : A \longrightarrow B$ is a function and $C \subset A$, then the function $x \longmapsto f(x)$, $x \in C$, from C to B is called the restriction of f to C and is denoted by $f|C$.

Page 20

Injective and onto functions

A function $f : A \longrightarrow B$ is said to be injective if $f(x) = f(y)$ only if $x = y$; f is said to be an onto function if $f(A) = B$.

Page 20

A function $f : A \longrightarrow B$ is said to be bijective if it is both injective and an onto function. A bijective function is also called a bijection or a one-one correspondence.

Page 21

Inverse function

If $f : A \longrightarrow B$ is a bijection, then there is a unique bijection $f^{-1} : B \longrightarrow A$ such that

$$f^{-1}(f(x)) = x, \quad x \in A \quad \text{and} \quad f(f^{-1}(y)) = y, \quad y \in B.$$

Page 22

Cartesian product

The Cartesian product of sets A_1 and A_2 is the set

$$A_1 \times A_2 = \{(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}.$$

Page 26

The first projection, $p_1 : A_1 \times A_2 \longrightarrow A_1$, is defined by

$$p_1(x_1, x_2) = x_1, \quad (x_1, x_2) \in A_1 \times A_2,$$

and the second projection, $p_2 : A_1 \times A_2 \longrightarrow A_2$, is defined by

$$p_2(x_1, x_2) = x_2, \quad (x_1, x_2) \in A_1 \times A_2.$$

Page 27

If $x \in A_1 \times A_2$, then

$$x = (p_1(x), p_2(x)).$$

Page 27

If A_1, A_2 are sets, $B_1 \subset A_1$, $B_2 \subset A_2$, and $p_1: A_1 \times A_2 \longrightarrow A_1$, $p_2: A_1 \times A_2 \longrightarrow A_2$ are the first and second projections, then

$$B_1 \times B_2 = p_1^{-1}(B_1) \cap p_2^{-1}(B_2).$$

SAQ 24, page 29

Countable sets

A set A is said to be countably infinite if there exists a bijection from $\mathbf{N}$ onto A, i.e. A is countably infinite if its elements can be enumerated as $a_1, a_2, \ldots, a_n, \ldots$.

Page 30

If a set is either finite or countably infinite, it is said to be a countable set. A set which is not countable is called uncountable.

Every infinite subset of $\mathbf{N}$ is countably infinite.

Remark, page 30

If A_1 and A_2 are countably infinite sets, then $A_1 \times A_2$ is a countably infinite set.

Example 23, page 31

The set $\mathbf{Q}$ of rational numbers is a countably infinite set.

Example 24, page 32

The set $\mathbf{R}$ of real numbers is an uncountable set.

Remark, page 32

The union of a countable family of countably infinite sets is a countably infinite set.

Proposition 13, page 32

Unit 2 Some Real Analysis

Intervals

Non-empty subsets of the set $\mathbf{R}$ of real numbers of the following forms are called intervals.

S page xiii

$$[a,b] = \{x \in \mathbf{R} : a \leq x \leq b\}, \qquad (a,b) = \{x \in \mathbf{R} : a < x < b\},$$
$$(a,b] = \{x \in \mathbf{R} : a < x \leq b\}, \qquad [a,b) = \{x \in \mathbf{R} : a \leq x < b\},$$
$$(-\infty, b] = \{x \in \mathbf{R} : x \leq b\}, \qquad [a, \infty) = \{x \in \mathbf{R} : x \geq a\},$$
$$(-\infty, b) = (x \in \mathbf{R} : x < b\}, \qquad (a, \infty) = (x \in \mathbf{R} : x > a\},$$
$$(-\infty, \infty) = \mathbf{R}.$$

Bounded sets

Let S be a non-empty set of real numbers. An upper bound for S is a number x such that $y \leq x$ for all y in S. If an upper bound for S exists, we say that S is bounded above. A lower bound for S is a number z such that $z \leq y$ for all y in S. If a lower bound for S exists, we say that S is bounded below. The set S is said to be bounded if it is bounded above and bounded below.

S *page 2*

Least upper bound

If S is a non-empty set of real numbers which is bounded above, then a number u is called a least upper bound for S if:

(a) u is an upper bound for S;

(b) $x \geq u$ for every upper bound x for S.

Definition 1.1.2, **S** *page 2*

Completeness axiom

A non-empty subset of $\mathbf{R}$ which is bounded above has a least upper bound.

Axiom 1.1.4, **S** *page 3*

A least upper bound for a non-empty subset S of $\mathbf{R}$ which is bounded above is unique and is denoted by sup S.

S *page 3*

Greatest lower bound

If S is a non-empty set of real numbers which is bounded below, then a number m is called a greatest lower bound for S if:

(a) m is a lower bound for S;

(b) $z \leq m$ for every lower bound z for S.

Note (vi), page 40

A non-empty subset S of $\mathbf{R}$ which is bounded below has a unique greatest lower bound denoted by inf S.

Proposition 1.1.5, **S** *page 3*

Archimedean property

The set $\mathbf{N}$ of natural numbers is not bounded above.

Proposition 1.1.6, **S** *page 4*

If ε is a positive real number, then there exists a positive integer N such that $\frac{1}{N} < \varepsilon$.

Note (ix), page 40

Rational, irrational density properties

Between two distinct real numbers there is a rational number and there is an irrational number.

Corollary 1.1.7, S page 4; Example 3, page 42

Triangle inequality

If $x, y \in \mathbf{R}$, then

$$|x + y| \leq |x| + |y|, \quad ||x| - |y|| \leq |x - y|.$$

Proposition 1.1.9, S page 5; Corollary 1.1.10, S page 5

Convergence of sequences

The sequence (s_n) of real numbers converges to the real number l if, given any real number $\varepsilon > 0$, there exists an integer N such that $|s_n - l| < \varepsilon$ for all $n \geq N$.

Other ways of writing '(s_n) converges to l' are '$s_n \longrightarrow l$ as $n \longrightarrow \infty$' and '$\lim_{n \to \infty} s_n = l$'.

Definition 1.2.2, S page 6

A convergent sequence has a unique limit.

Proposition 1.2.3, S page 8

Monotonic sequences

Let (s_n) be a sequence of real numbers. The sequence (s_n) is monotonic increasing if $s_n \leq s_{n+1}$ for all n in $\mathbf{N}$; it is monotonic decreasing if $s_{n+1} \leq s_n$ for all n in $\mathbf{N}$; it is monotonic if it has either of these properties.

Definition 1.2.5, S page 8

Every bounded monotonic sequence converges.

Theorem 1.2.6, S page 9

Continuity

A function $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for every x such that $|x - a| < \delta$.

Definition 1.4.3', S page 14

A function $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous if it is continuous at a for all a in $\mathbf{R}$.

Definition 1.4.4, S page 15

Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous at a. Then $|f|: \mathbf{R} \longrightarrow \mathbf{R}$ given by $|f|(x) = |f(x)|$, $x \in \mathbf{R}$, is continuous at a. If $k \in \mathbf{R}$ then $kf: \mathbf{R} \longrightarrow \mathbf{R}$ given by $(kf)(x) = kf(x)$, $x \in \mathbf{R}$, is continuous at a. If $f(x) \neq 0$ for all x, then $\frac{1}{f}: \mathbf{R} \longrightarrow \mathbf{R}$ given by $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$, $x \in \mathbf{R}$, is continuous at a.

SAQ 20 (i), (ii), (v), pages 59, 60

If $f: \mathbf{R} \longrightarrow \mathbf{R}$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ are continuous at a, then $f + g: \mathbf{R} \longrightarrow \mathbf{R}$ and $fg: \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x)g(x), \quad x \in \mathbf{R},$$

are continuous at a.

SAQ 20 (iii), (iv), page 60

Unit 3 Metric Spaces and Continuity

Metric spaces

A metric space $M = \{A, d\}$ consists of a non-empty set A together with a map $d: A \times A \longrightarrow \mathbf{R}$, called a metric on A, which satisfies the following three conditions:

(M1) $d(x, y) \geq 0$ for all x, y in A, and $d(x, y) = 0$ if and only if $x = y$;

(M2) $d(x, y) = d(y, x)$ for all x, y in A;

(M3) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z in A.

Definition 2.1.2, S page 21

Metric subspaces

If $M = \{A, d\}$ is a metric space and H is a non-empty subset of A, then $d_H = d|H \times H$ is a metric on H; the metric space $\{H, d_H\}$ is called a subspace of M and d_H is called the metric on H induced by d.

Example 2.2.5, S page 24

Product spaces Let $\{A, d\}$, $\{A', d'\}$ be metric spaces and let d_1, d_2 and d_∞ be the real-valued functions on $A \times A'$ defined as follows: if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $A \times A'$, then

$$d_1(x, y) = d(x_1, y_1) + d'(x_2, y_2),$$
$$d_2(x, y) = [\{d(x_1, y_1)\}^2 + \{d'(x_2, y_2)\}^2]^{1/2},$$
$$d_\infty(x, y) = \text{Max}\{d(x_1, y_1), d'(x_2, y_2)\}.$$

Then d_1, d_2 and d_∞ are metrics on $A \times A'$, each of which is called a product metric.

Example 2.2.7, S page 26

Euclidean metric If $H \subset \mathbf{R}^n$, then the Euclidean metric on H is the metric d on H defined as follows: if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in H$ then

$$d(x, y) = \sqrt{\left(\sum_{i=1}^{n} (x_i - y_i)^2\right)}.$$

Note (ii), page 72

Bounded sets A subset S of a metric space $\{A, d\}$ is said to be bounded if there exist some a in A and a real number K such that $d(x, a) \leq K$ for all x in S.

Definition 2.2.12, S page 29

A subset S of a metric space $\{A, d\}$ is bounded if and only if there exists a real number L such that $d(x, y) \leq L$ for all x, y in S.

Note (iv), page 78

Diameter If S is a non-empty bounded subset of a metric space $\{A, d\}$ then the diameter, diam S, of S is $\sup D$, where D is the set of all real numbers $d(x, y)$, $x, y \in S$.

Definition 2.2.13, S page 30

Bounded maps If $f: X \longrightarrow M$ is a map into a metric space M, then f is bounded if the subset $f(X)$ of M is bounded.

Definition 2.2.14, S page 30

Open balls If $M = \{A, d\}$ is a metric space, $a \in A$ and ε is a strictly positive real number, then the open ε-ball neighbourhood of a in M is the set

$$B_\varepsilon(a; d) = \{x \in A : d(x, a) < \varepsilon\}.$$

If the context makes the metric clear, $B_\varepsilon(a)$ may be written instead of $B_\varepsilon(a; d)$. The sets $B_\varepsilon(a)$ are often called open balls.

Definition 2.3.1, S page 33

Open sets A subset U of a metric space M is open in M if, for each y in U, there exists $\varepsilon(y) > 0$ such that $B_{\varepsilon(y)}(y) \subset U$. If $M = \{A, d\}$ and U is open in M, we also say that U is d-open.

Definition 2.3.8, S page 34

Every open ball in a metric space M is open in M.

Lemma 2.3.7 and Examples 2.3.9, S page 34

If $U_1, U_2, \ldots, U_m$ are open in a metric space M, then $\bigcap_{i=1}^{m} U_i$ is open in M.

Proposition 2.3.17, S page 38

The union of a collection of sets open in a metric space M is open in M.

Proposition 2.3.19, S page 38

A subset of a metric space M is open in M if and only if it is a union of open balls.

SAQ 18, page 83

If x and y are distinct points of a metric space M, then there are disjoint sets V and W open in M such that $x \in V$ and $y \in W$.

SAQ 19, page 83

Continuity Let $M_1 = \{A_1, d_1\}$, $M_2 = \{A_2, d_2\}$ be metric spaces and let $f: A_1 \longrightarrow A_2$ be a map. The map f is said to be (d_1, d_2)-continuous at a point a in A_1 if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta$ implies $d_2(f(x), f(a)) < \varepsilon$. If f is (d_1, d_2)-continuous at every a in A_1, we say that f is (d_1, d_2)-continuous, or that $f: M_1 \longrightarrow M_2$ is continuous.

Definition 2.1.3, S page 22

The map f is (d_1, d_2)-continuous at a in A_1 if and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(a; d_1)) \subset B_\varepsilon(f(a); d_2)$.

Note (iii), page 81

The map $f: M_1 \longrightarrow M_2$ is continuous if and only if for every set U open in M_2, $f^{-1}(U)$ is open in M_1.

Proposition 2.3.13, S page 36

Equivalent metrics

Metrics d_1 and d_2 on a set A are topologically equivalent if and only if the following condition is satisfied:

(TE) For each a in A and each $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$B_{\delta_1}(a; d_1) \subset B_\varepsilon(a; d_2) \quad \text{and} \quad B_{\delta_2}(a; d_2) \subset B_\varepsilon(a; d_1).$$

Note (ii), page 85

Metrics d_1 and d_2 on a set A are topologically equivalent if and only if the following condition is satisfied: a subset U of A is d_1-open if and only if it is d_2-open.

Proposition 2.4.2, **S** *page 39*

Homeomorphisms

If $M = \{A, d\}$ and $M' = \{A', d'\}$ are metric spaces, then a topological equivalence, or homeomorphism, is a one-one correspondence $f: A \longrightarrow A'$ such that $f: M \longrightarrow M'$ and $f^{-1}: M' \longrightarrow M$ are continuous.

Definition 2.4.7, **S** *page 41*

Unit 4 Topological Spaces I

Topological space, topology

A topological space $T = \{A, \mathscr{T}\}$ consists of a non-empty set A together with a fixed collection $\mathscr{T}$ of subsets of A satisfying:

(T1) $A, \varnothing \in \mathscr{T}$;

(T2) the intersection of two sets in $\mathscr{T}$ is in $\mathscr{T}$;

(T3) the union of each collection of sets in $\mathscr{T}$ is in $\mathscr{T}$.

The collection $\mathscr{T}$ is called a topology on A.

Definition 3.1.1, **S** *page 45*

Open sets

If $T = \{A, \mathscr{T}\}$ is a topological space and U is a subset of A, then U is said to be an open set of T, or to be open in T, if and only if $U \in \mathscr{T}$.

S *page 45*

A subset V of a topological space T is an open set of T if and only if for each x in V there exists an open set V_x of T such that $x \in V_x \subset V$.

SAQ 4, page 97

Comparison of topologies

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on a set A. If $\mathscr{T}_2 \subset \mathscr{T}_1$ then $\mathscr{T}_1$ is said to be finer than $\mathscr{T}_2$ and $\mathscr{T}_2$ is said to be coarser than $\mathscr{T}_1$. If $\mathscr{T}_2 \subset \mathscr{T}_1$ and $\mathscr{T}_2 \neq \mathscr{T}_1$ then $\mathscr{T}_1$ is said to be strictly finer than $\mathscr{T}_2$ and $\mathscr{T}_2$ is said to be strictly coarser than $\mathscr{T}_1$. If $\mathscr{T}_1$ is neither finer nor coarser than $\mathscr{T}_2$, then $\mathscr{T}_1$ and $\mathscr{T}_2$ are said to be non-comparable.

S *page 47; Note (iii), page 95*

Metric spaces

A metric d on a set A gives rise to a topology $\mathscr{T}$ on A, where $\mathscr{T}$ is the collection of d-open subsets of A (i.e. subsets of A which are open in the metric space $\{A, d\}$).

Example 3.1.5, **S** *page 46*

Metrics d_1 and d_2 on a set A give rise to the same topology on A if and only if they are topologically equivalent.

Note (ii), pages 94, 95

Metrizable spaces

A topological space $\{A, \mathscr{T}\}$ is called a metrizable space if there exists a metric d on A which gives rise to the topology $\mathscr{T}$ on A.

S *page 46*

Topology of $\mathbf{R}^n$

The Euclidean metric on $\mathbf{R}^n$ (or any topologically equivalent metric) gives rise to the usual topology on $\mathbf{R}^n$.

Note (ii), page 95

Continuity

Given two topological spaces $T_1 = \{A_1, \mathscr{T}_1\}$ and $T_2 = \{A_2, \mathscr{T}_2\}$, and a map $f: A_1 \longrightarrow A_2$, we say that f is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous, or that $f: T_1 \longrightarrow T_2$ is continuous, if, for every $U \in \mathscr{T}_2$, $f^{-1}(U) \in \mathscr{T}_1$.

Definition 3.1.2, **S** *page 45*

If T_1, T_2, T_3 are topological spaces and $f: T_1 \longrightarrow T_2$, $g: T_2 \longrightarrow T_3$ are continuous maps, then $g \circ f: T_1 \longrightarrow T_3$ is continuous.

Proposition 3.1.8, **S** *page 48*

The identity map on a topological space is continuous; a constant map between topological spaces is continuous.

Proposition 3.1.10, **S** *page 48; Notes (v), (vi), page 100*

If, for $i = 1, 2$, the metric d_i on A_i gives rise to the topology $\mathscr{T}_i$ on A_i, and $f: A_1 \longrightarrow A_2$ is (d_1, d_2)-continuous, then f is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous.

Proposition 3.1.11, **S** *page 48*

Bases

Given a topological space $T = \{A, \mathscr{T}\}$, a basis for the topology $\mathscr{T}$ is a subcollection $\mathscr{B}$ of $\mathscr{T}$ such that every set in $\mathscr{T}$ is a union of sets from $\mathscr{B}$.

Definition 3.2.1, **S** *page 49*

To show that $f: T_1 \longrightarrow T_2$ is continuous, it is enough to prove that, for each set B in some basis for the topology of T_2, $f^{-1}(B)$ is open in T_1. *Application 3.2.5, S page 50*

A synthetic basis for a set A is a collection $\mathscr{B}$ of subsets of A such that: *Definition 3.2.2, S page 49*

(B1) A is a union of sets from $\mathscr{B}$;

(B2) if $B_1, B_2 \in \mathscr{B}$, then $B_1 \cap B_2$ is a union of sets from $\mathscr{B}$.

A collection $\mathscr{B}$ of subsets of a set A which satisfies (B1) is a synthetic basis for A if and only if the following condition is satisfied: *Note (iii), page 103*

(B2′) If $B_1, B_2 \in \mathscr{B}$, then for each x in $B_1 \cap B_2$ there exists B_x in $\mathscr{B}$ such that $x \in B_x \subset B_1 \cap B_2$.

If $\mathscr{B}$ is a synthetic basis for a non-empty set A and $\mathscr{T} = \{U \in 2^A: U \text{ is a union of sets from } \mathscr{B}\}$, then $\mathscr{T}$ is a topology on A called the topology arising from $\mathscr{B}$. *Proposition 3.2.3, S page 49*

If $\mathscr{T}$ is the topology on a set A arising from the synthetic basis $\mathscr{B}$ for A, then $\mathscr{B}$ is a basis for $\mathscr{T}$. If $\mathscr{T}$ is a topology on a set A and $\mathscr{B}$ is a basis for $\mathscr{T}$, then $\mathscr{B}$ is a synthetic basis for A, and the topology arising from $\mathscr{B}$ is $\mathscr{T}$. *Proposition 3.2.4, S page 49*

Given a synthetic basis $\mathscr{B}$ on a non-empty set A, the topology arising from $\mathscr{B}$ is called 'the topology on A with $\mathscr{B}$ as basis'. *Application 3.2.6, S page 50*

Subspaces

If $T = \{A, \mathscr{T}\}$ is a topological space and H is a non-empty subset of A, then the collection $\mathscr{T}_H = \{U \cap H: U \in \mathscr{T}\}$ is a topology on H, called the relative, induced, or subspace topology on H. The topological space $\{H, \mathscr{T}_H\}$ is called a subspace of T. *Definition 3.4.1, S page 51*

If T_1, T_2 are topological spaces, $f: T_1 \longrightarrow T_2$ is continuous, and H_1, H_2 are subspaces of T_1, T_2 respectively, such that $f(H_1) \subset H_2$, then $g: H_1 \longrightarrow H_2$ given by $g(x) = f(x)$, $x \in H_1$, is continuous. *Note (v), page 106*

If d is a metric on a set A which gives rise to the topology $\mathscr{T}$ on A, $H \subset A$, and d_H is the metric on H induced by d, then d_H gives rise to the topology $\mathscr{T}_H$ induced on H by $\mathscr{T}$. *SAQ 17, page 108*

Topological products

If $T_1 = \{A_1, \mathscr{T}_1\}$, $T_2 = \{A_2, \mathscr{T}_2\}$ are topological spaces, the product topology $\mathscr{T}$ on $A_1 \times A_1$ is the topology with basis $\{U_1 \times U_2: U_1 \in \mathscr{T}_1, U_2 \in \mathscr{T}_2\}$. The space $T_1 \times T_2 = \{A_1 \times A_2, \mathscr{T}\}$ is called the topological product of T_1 and T_2. *Definition 3.5.1, S page 54*

If $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for the topologies of T_1 and T_2, then a basis for the topology of $T_1 \times T_2$ consists of all sets of the form $B_1 \times B_2$, where $B_1 \in \mathscr{B}_1$, $B_2 \in \mathscr{B}_2$. *Note (ii), page 109*

The projections $p_1: T_1 \times T_2 \longrightarrow T_1$ and $p_2: T_1 \times T_2 \longrightarrow T_2$, of a topological product onto its factors, are continuous. If $f: T' \longrightarrow T_1 \times T_2$ is a map from a topological space T' to the topological product $T_1 \times T_2$, then f is continuous if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous. *Propositions 3.5.3, 3.5.4, S page 55*

If $\mathbf{R}$ has its usual topology, then the product topology on $\mathbf{R}^n = \mathbf{R} \times \ldots \times \mathbf{R}$ (n factors) is the usual topology on $\mathbf{R}^n$. *S page 56*

Unit 5 Topological Spaces II

Closed sets

A subset V of a topological space T is closed in T if $T - V$ is open in T. *Definition 3.7.1, S page 59*

Let T be a topological space. The empty set $\varnothing$ and T itself are closed in T; the union of two sets closed in T is closed in T; the intersection of each family of sets closed in T is closed in T. *Proposition 3.7.4, S page 60*

Let T be a topological space and let T' be a subspace of T. Then $V \subset T'$ is closed in T' if and only if $V = T' \cap W$ for some W closed in T. *Proposition 3.7.6, S page 60*

A map $f: T_1 \longrightarrow T_2$ of topological spaces is continuous if and only if $f^{-1}(V)$ is closed in T_1 for every V closed in T_2. *Proposition 3.7.8, S page 61*

A subset E of a topological space T is closed in T if and only if for each point x of $T - E$ there exists an open set V_x such that $x \in V_x$ and $V_x \cap E = \varnothing$. *SAQ 3, page 8*

Limit point

A point x of a topological space T is a limit point of a subset H of T if every open set containing x contains some point of H other than x. *Definition 3.7.9, S page 61*

Closure

The closure, $Cl(H)$, of a subset H of a topological space T is the union of H and the set of all limit points of H in T.

Definition 3.7.11, S page 61

Let H be a subset of a topological space T. Then:

(a) a point x of T is in $Cl(H)$ if and only if every open set containing x contains a point of H;

(b) $Cl(H)$ is closed in T and is the smallest closed set containing H, in the sense that if E is closed and $H \subset E$ then $Cl(H) \subset E$;

(c) H is closed in T if and only if $Cl(H) = H$.

Propositions 3.7.12, 3.7.15, 3.7.16, S pages 61, 62

If $H_1, H_2, \ldots, H_n$ are subsets of a topological space T, then

$$Cl(H_1 \cup H_2 \cup \ldots \cup H_n) = Cl(H_1) \cup Cl(H_2) \cup \ldots \cup Cl(H_n).$$

Proposition 3.7.18, S page 63

Distance of a point from a set

If $M = \{A, d\}$ is a metric space, H is a non-empty subset of A, and $x \in A$, then the distance of x from H is defined to be

$$d(x, H) = \inf\{d(x, y) : y \in H\}.$$

Example 7, page 11

The closure of H in the metric space M is

$$Cl(H) = \{x \in A : d(x, H) = 0\}.$$

Interior

The interior, $\operatorname{Int} H$, of a subset H of a topological space T is the union of all subsets of H which are open in T.

Definition 3.7.24, S page 64

Let H be a subset of a topological space T. Then:

(a) $\operatorname{Int} H$ is the largest subset of H which is open in T;

(b) H is open in T if and only if $H = \operatorname{Int} H$;

(c) $Cl(T - H) = T - \operatorname{Int} H$.

Propositions 1, 2, 4, pages 13, 14

If H is a subset of a metric space $M = \{A, d\}$ then

$$\operatorname{Int} H = \{x \in A : d(x, A - H) > 0\}.$$

SAQ 16, page 16

Boundary

The boundary, $b(H)$, of a subset H of a topological space T is $Cl(H) \cap Cl(T - H)$.

Definition 3.7.31, S page 64

If H is a subset of a topological space T, then $\operatorname{Int} H$ and $b(H)$ are disjoint and $(\operatorname{Int} H) \cup b(H) = Cl(H)$.

Proposition 5, page 15

Dense sets

A subset H is dense in a topological space T if $Cl(H) = T$.

Definition 3.7.20, S page 63

A subset H of a topological space T is dense in T if and only if $V \cap H \neq \varnothing$ for every non-empty open set V of T.

Note (i), page 17

Nowhere-dense sets

A subset H of a topological space T is nowhere-dense in T if $\operatorname{Int}(Cl(H)) = \varnothing$.

Definition 3.7.26, S page 64

A subset H of a topological space T is nowhere-dense in T if and only if $T - Cl(H)$ is dense in T.

Proposition 3.7.28, S page 64

Hausdorff spaces

A topological space T is a Hausdorff space if, whenever x, y are distinct points of T, there exist disjoint sets U, V, open in T, containing x, y respectively.

Definition 4.2.1, S page 73

Every metric space is a Hausdorff space.

Note (iii), page 20

Every subspace of a Hausdorff space is Hausdorff; the topological product of two Hausdorff spaces is Hausforff.

Proposition 4.2.4, S page 73

Homeomorphisms

A homeomorphism $f: T_1 \longrightarrow T_2$ of topological spaces is a bijection such that f and f^{-1} are both continuous.

Definition 3.6.1, S page 56

If $f: T_1 \longrightarrow T_2$ is a bijection where T_1 and T_2 are topological spaces, then the following statements are equivalent:

(a) f is a homeomorphism;

(b) a subset U of T_1 is open in T_1 if and only if $f(U)$ is open in T_2.

Proposition 8, page 22

Topological property

If P is some property which makes sense for all topological spaces, and if whenever a topological space T has property P so has each topological space homeomorphic with T, then P is called a topological property.

S page 59

Unit 6 Compactness

Cover, subcover

A cover of a set A is a collection $\mathscr{U}$ of sets such that $A \subset \bigcup_{U \in \mathscr{U}} U$. *Definition 5.2.1, S page 77*

A subcover of a given cover $\mathscr{U}$ of A is a subcollection $\mathscr{V} \subset \mathscr{U}$ which forms a cover of A.

A cover $\mathscr{U}$ is finite if there are only finitely many sets in the collection $\mathscr{U}$.

If A is a subspace of a topological space T, and if $\mathscr{U}$ is a cover of A in which each set $U \in \mathscr{U}$ is open in T, then $\mathscr{U}$ is an open cover of A.

Compact spaces

A topological space T is compact if every open cover of T has a finite subcover. *Definition 5.2.2, S page 77*

A subspace T of a topological space S is a compact space if and only if each cover of T by sets open in S contains a finite subcover of T. *Remark 5.2.3(a), S page 78*

Compact subsets

If $T \subset S$ where S is a topological space, then T is said to be a compact subset of S if T with the induced topology as a subspace of S is a compact space. *Note (ii), page 33*

Heine–Borel Theorem

Every bounded closed interval $[a, b]$ in $\mathbf{R}$ is compact. *Theorem 5.3.1, S page 79*

Bolzano–Weierstrass Theorem

If S is an infinite subset of the bounded closed interval $[a, b]$ in $\mathbf{R}$, then S has a limit point in $[a, b]$. *Proposition 2, page 37*

Results on compactness

A compact subspace of a metric space is bounded. *Proposition 5.4.1, S page 82*

A compact subspace of a Hausdorff space is closed. *Proposition 5.4.2, S page 82*

If $f: T_1 \longrightarrow T_2$ is a continuous map of topological spaces and T_1 is compact, then $f(T_1)$ is compact. *Proposition 5.5.1, S page 83*

Compactness is a topological property. *Corollary 5.5.2, S page 83*

A continuous map from a compact space to a metric space is bounded. In particular, if $f: C \longrightarrow \mathbf{R}$ is continuous and C is a compact space, then f is bounded on C and attains its bounds. *Corollaries 5.5.3, 5.5.4, S pages 83, 84*

A closed subspace of a compact space is compact. *Proposition 5.6.1, S page 84*

A topological product $T_1 \times T_2$ is compact if and only if T_1 and T_2 are both compact. *Theorem 5.6.2, S page 84*

If $f: T_1 \longrightarrow T_2$ is a continuous bijection where T_1 is a compact space and T_2 is a Hausdorff space, then f is a homeomorphism. *Theorem 5.9.1, S page 88*

Compactness in $\mathbf{R}^n$

A subspace of $\mathbf{R}^n$ is compact if and only if it is closed and bounded (with respect to the Euclidean metric). *Theorem 5.7.1, S page 86; Note (ii), page 44*

Uniform continuity

A map $f: M_1 \longrightarrow M_2$ of metric spaces (with metrics d_1, d_2) is uniformly continuous on M_1 if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_2(f(x), f(y)) < \varepsilon$ for all x, y in M_1 satisfying $d_1(x, y) < \delta$. *Definition 5.8.1, S page 87*

If $f: M_1 \longrightarrow M_2$ is a continuous map of metric spaces where M_1 is compact, then f is uniformly continuous on M_1. *Proposition 5.8.2, S page 87*

Unit 7 Connectedness

Connected spaces

A topological space T is connected if there exists no continuous map from T onto a discrete two-point space. *Definition 6.2.1, S page 94*

Partitions

A partition $A|B$ of a topological space T is a pair of disjoint non-empty subsets A, B of T such that $A \cup B = T$ and both A and B are open in T. *Definition 6.2.2, S page 94*

Characterizations

The following statements about a topological space T are equivalent: *Proposition 6.2.3, S page 94*

(1) T is connected;

(2) T admits no partition;

(3) the only subsets of T which are both open and closed in T are $\varnothing$ and T.

A subspace of $\mathbf{R}$ is connected if and only if it is an interval. *Theorems 6.2.7, 6.2.8, S page 96*

Connectedness properties

If T_1 and T_2 are topological spaces with T_1 connected, and $f: T_1 \longrightarrow T_2$ is continuous, then $f(T_1)$ is connected. *Proposition 6.2.10, S page 97*

Connectedness is a topological property. *Corollary 6.2.11, S page 97*

If T is a connected space and $f: T \longrightarrow \mathbf{R}$ is continuous, then $f(T)$ is an interval. *Corollary 6.2.13, S page 98*

If $(C_i)_{i \in I}$ is a family of connected subspaces of a space T, and B is a connected subspace of T such that $B \cap C_i \neq \varnothing$ for each i in I, then $B \cup \left(\bigcup_{i \in I} C_i \right)$ is connected. *Corollary 6.2.16, S page 98*

A product space $T_1 \times T_2$ is connected if and only if T_1 and T_2 are connected. *Proposition 6.2.17, S page 98*

If H is a connected subspace of a topological space T, and if $H \subset K \subset Cl(H)$, then K is connected. *Proposition 6.2.18, S page 99*

Paths

Given two points a, b in a topological space T, a path from a to b in T is a continuous map $f: [0,1] \longrightarrow T$ such that $f(0) = a$ and $f(1) = b$. Such an f is called a path joining a and b. *Definition 6.3.1, S page 100*

The relation $\equiv$, defined on a topological space T by putting $a \equiv b$ if there is a path in T joining a and b, is an equivalence relation. *Proposition 2, page 64*

If $a, b \in \mathbf{R}^n$, then $f: [0,1] \longrightarrow \mathbf{R}^n$ given by *Proposition 3, page 65*

$$f(t) = (1-t)a + tb, \quad t \in [0,1],$$

is a path in $\mathbf{R}^n$ joining a and b.

Path-connected spaces

A topological space T is path-connected if every pair of points in T can be joined by a path in T. *Definition 6.3.2, S page 100*

Every path-connected space is connected. *Proposition 6.4.1, S page 102*

There exist connected spaces which are not path-connected. *Example 6.3.5, S page 101*

A connected open subspace of $\mathbf{R}^n$ is path-connected. *Proposition 6.4.2, S page 102*

If K is a countable subset of $\mathbf{R}^n$, $n \geq 2$, then the subspace $\mathbf{R}^n - K$ is path-connected. *Example 8, page 66*

Convex sets

A subset C of $\mathbf{R}^n$ is convex if every pair of points in C can be joined by a line segment in C. Every convex subspace of $\mathbf{R}^n$ is path-connected. *Example 7, page 65*

Components

The components of a topological space T are the equivalence classes with respect to the following equivalence relation $\sim$ on T: $x \sim y$ if there exists a connected subspace C of T such that $x \in C$ and $y \in C$. *S page 103*

The components of a topological space T are the maximal connected subspaces of T, in the following sense: C is a component of T if and only if C is connected and there is no connected subspace of T which contains C as a proper subset. *Note (ii), page 70*

The components of a topological space T form a disjoint covering of T by closed sets. *Note (ix), page 72*

If topological spaces T_1 and T_2 are homeomorphic, then there is a one-one correspondence between the sets of components of T_1 and T_2 respectively. *Theorem 6.5.4, S page 104*

Unit 8 Completeness

Convergence

A sequence (x_n) in a metric space M converges to a point x of M if, given $\varepsilon > 0$, there exists N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. *Definition 4.1.1*, **S** *page 72*

Cauchy sequences

A sequence (x_n) in a metric space with metric d is a Cauchy sequence if, given $\varepsilon > 0$, there exists N such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$. *Definition 7.2.3*, **S** *page 109*

A convergent sequence in a metric space M is a Cauchy sequence in M. *Proposition 2, page 84*

There exist metric spaces which contain non-convergent Cauchy sequences. *Example 2, page 84*

Convergence and closure

If C is a subset of a metric space M, then a point x of M is in $Cl(C)$ if and only if there exists a sequence in C converging to x. *Lemma 7.2.2*, **S** *page 108*

ε-nets

Given $\varepsilon > 0$, an ε-net for a metric space M is a subset S of M such that $M \subset \bigcup_{x \in S} B_\varepsilon(x)$. *Definition 7.2.8*, **S** *page 110*

Lebesgue number

Given an open cover $\mathscr{U}$ of a metric space M, a positive real number ε is called a Lebesgue number for $\mathscr{U}$ if, for each x in M, there exists a set $U(x)$ in $\mathscr{U}$ such that $B_\varepsilon(x) \subset U(x)$. *Definition 7.2.11*, **S** *page 111*

Function spaces

If $\mathscr{B}[a, b], \mathbf{R})$ denotes the set of bounded real-valued functions on a closed interval $[a, b]$, then d_∞, defined by *Page 91*

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|, \quad f, g \in \mathscr{B}([a, b], \mathbf{R}),$$

is a metric on $\mathscr{B}([a, b], \mathbf{R})$. We put $\mathbf{B}[a, b] = \{\mathscr{B}([a, b], \mathbf{R}), d_\infty\}$.

The metric induced by d_∞ on the subset $\mathscr{C}[a, b]$ of $\mathscr{B}([a, b], \mathbf{R})$ which consists of the continuous real-valued functions on $[a, b]$ is also denoted by d_∞; and $\mathbf{C}[a, b] = \{\mathscr{C}[a, b], d_\infty\}$.

If $f \in \mathscr{C}[a, b]$ and ε is a positive real number, then in $\mathbf{C}[a, b]$, $B_\varepsilon(f)$ consists of all those functions g in $\mathscr{C}[a, b]$ such that *Proposition 4, page 92*

$$|f(x) - g(x)| < \varepsilon$$

for all x in $[a, b]$.

A sequence (f_n) converges in $\mathbf{B}[a, b]$ to f if and only if the following condition is satisfied: for each $\varepsilon > 0$, there exists a positive integer N such that *Proposition 5, page 93*

$$|f(x) - f_n(x)| < \varepsilon$$

for all $n \geq N$ and all x in $[a, b]$.

The subspace $\mathbf{C}[a, b]$ is closed in $\mathbf{B}[a, b]$. *Corollary, page 96*

A sequence (f_n) is a Cauchy sequence in $\mathbf{B}[a, b]$ if and only if for each $\varepsilon > 0$ there exists a positive integer N such that $|f_n(x) - f_m(x)| < \varepsilon$ for all x in $[a, b]$ and all $m, n \geq N$. *Proposition 7, page 96*

Complete metric spaces

A metric space M is complete if every Cauchy sequence in M converges to a point in M. *Definition 9.2.1*, **S** *page 123*

A complete subspace of a metric space M is closed in M. *Proposition 9.2.3(a)* **S** *page 123*

A closed subspace of a complete metric space is complete. *Proposition 9.2.3(b)* **S** *page 124*

A compact metric space is complete. *Proposition 9.2.4*, **S** *page 124*

For each positive integer n, $\mathbf{R}^n$ is complete. *Example 9.2.6*, **S** *page 124*

The function spaces $\mathbf{B}[a, b]$ and $\mathbf{C}[a, b]$ are complete. *Example 10, page page 100*

Completeness is not a topological property of metric spaces. *Note, page 100*

Produced in Great Britain by Speedlith Photo Litho Limited, Manchester